Chapter 2

Metric Spaces

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A metric space is a mathematical object in which the distance between two points is meaningful. Metric spaces constitute an important class of topological spaces. We introduce metric spaces and give some examples in Section 1. In Section 2 open and closed sets are introduced and we discuss how to use them to describe the convergence of sequences and the continuity of functions. Relevant notions such as the boundary points, closure and interior of a set are discussed. Compact sets are introduced in Section 3 where the equivalence between the Bolzano-Weierstrass formulation and the finite cover property is established. In Sections 4 and 5 we turn to complete metric spaces and the contraction mapping principle. As an application of the latter, we study the initial value problem for differential equations and establish the fundamental existence and uniqueness theorem in Section 6.

2.1 Metric Spaces

Throughout this chapter X always denotes a non-empty set. We would like to define a concept of distance which assigns a positive number to every two distinct points in X. In analysis the name metric is used instead of distance. (But "d" not "m" is used in notation. I have no idea why it is so.) A **metric** on X is a function from $X \times X$ to $[0, \infty)$ which satisfies the following three conditions: $\forall x, y, z \in X$,

M1. $d(x, y) \geq 0$ and equality holds if and only if $x = y$,

M2. $d(x, y) = d(y, x)$, and

M3. $d(x, y) \leq d(x, z) + d(z, y)$.

The last condition, the triangle inequality, is a key property of a metric. M2 and M3 together imply another form of triangle inequality,

$$|d(x,y) - d(x,z)| \le d(y,z).$$

The pair (X, d) is called a **metric space**. Let $x \in X$ and $r > 0$, the **metric ball** or simply the ball $B_r(x)$ is the set $\{y \in X : d(y,x) < r\}$.

Here are some examples of metric spaces.

Example 2.1. Let $\mathbb{R}$ be the set of all real numbers. For $x, y \in \mathbb{R}$, we define $d(x,y) = |x-y|$ where $|x|$ denotes the absolute value of x. It is easily seen that $d(\cdot,\cdot)$ satisfies M1-M3 above and so it defines a metric. In particular, M3 reduces to the usual triangle inequality. Thus $(\mathbb{R}, d)$ is a metric space. From now on whenever we talk about $\mathbb{R}$, it is understood that it is a metric space endowed with this metric.

Example 2.2. More generally, let $\mathbb{R}^n$ be the n-dimensional real vector space consisting of all n-tuples $x = (x_1, \dots, x_n)$, $x_j \in \mathbb{R}$, $j = 1, \dots, n$. For $x, y \in \mathbb{R}^n$, introduce the **Euclidean metric**

$$\begin{aligned} d_2(x,y) &= \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2} \\ &= \Big(\sum_{j=1}^n (x_j - y_j)^2\Big)^{\frac{1}{2}}. \end{aligned}$$

It reduces to Example 1 when $n = 1$. Apparently, M3 and M2 are fulfilled. To verify the triangle inequality, letting $u = x - z$ and $v = x - y$, M3 becomes

$$\Big(\sum_1^n (u_j + v_j)^2\Big)^{1/2} \le \Big(\sum_1^n u_j^2\Big)^{1/2} + \Big(\sum_1^n v_j^2\Big)^{1/2}.$$

Taking square, we see that it follows from Cauchy-Schwarz inequality

$$\Big|\sum_1^n u_j v_j\Big| \le \Big(\sum_1^n u_j^2\Big)^{1/2}\Big(\sum_1^n v_j^2\Big)^{1/2}.$$

In case you do not recall its proof, look up a book. We need to use mathematics you learned in all previous years. Take this as a chance to refresh them.

Example 2.3. It is possible to have more than one metrics on a set. Again consider $\mathbb{R}^n$. Instead of the Euclidean metric, we define

$$d_1(x,y) = \sum_{j=1}^n |x_j - y_j|$$

and

$$d_\infty(x,y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}.$$

It is not hard to verify that d_1 and d_∞ are also metrics on $\mathbb{R}^n$. We denote the metric balls in the Euclidean, d_1 and d_∞ metrics by $B_r(x)$, $B_r^1(x)$ and $B_r^\infty(x)$ respectively. $B_r(x)$ is the standard ball of radius r centered at x and $B_r^\infty(x)$ is the cube of length r centered at x. I let you draw and tell me what $B_r^1(x)$ looks like.

Example 2.4. Let $C[a,b]$ be the real vector space of all continuous, real-valued functions on $[a,b]$. For $f,g \in C[a,b]$, define

$$d_\infty(f,g) = \|f-g\|_\infty \equiv \max\left\{|f(x)-g(x)| : x \in [a,b]\right\}.$$

It is easily checked that d_∞ is a metric on $C[a,b]$. The metric ball $B_r(f)$ in the uniform metric consists of all continuous functions sitting inside the "tube"

$$\left\{(x,y) : |y-f(x)| < r, x \in [a,b]\right\}.$$

Another metric defined on $C[a,b]$ is given by

$$d_1(f,g) = \int_a^b |f-g|.$$

It is straightforward to verify M1-M3 are satisfied. In Section 1.5 we encountered the L^2-distance. Indeed,

$$d_2(f,g) = \sqrt{\int_a^b |f-g|^2},$$

really defines a metric on $C[a,b]$. The verification of M3 to similar to what we did in Example 2.2, but Cauchy-Schwarz inequality is now in integral form

$$\int_a^b |fg| \le \sqrt{\int_a^b f^2}\sqrt{\int_a^b g^2}.$$

In passing we point out same notations such as d_1 and d_2 have been used to denote different metrics. They arise in quite different context though. It should not cause confusion.

Example 2.5. Let $R[a,b]$ be the vector space of all Riemann integrable functions on $[a,b]$ and consider $d_1(f,g)$ as defined in the previous example. One can show that M2 and M3 are satisfied, but not M1. In fact,

$$\int_a^b |f-g| = 0$$

does not imply that f is equal to g. It tells us they differ on a set of measure zero. This happens, for instance, they are equal except at finitely many points. To construct a metric space out of d_1, we introduce a relation on $R[a,b]$ by setting $f \sim g$ if and only if f and g differ on a set of measure zero. It is routine to verify that $\sim$ is an equivalence relation.

Let $\widetilde{R}[a,b]$ be the equivalence classes of $R[a,b]$ under this relation. We define a metric on $\widetilde{R}[a,b]$ by, $\forall \overline{f}, \overline{g} \in \widetilde{R}[a,b]$,

$$\widetilde{d_1}(\overline{f},\overline{g}) = d_1(f,g), \quad f \in \overline{f},\ g \in \overline{g}.$$

Then $(\widetilde{R}[a,b], \widetilde{d_1})$ forms a metric space. I let you verify that $\widetilde{d_1}$ is well-defined, that is, it is independent of the choices of f and g, and is a metric on $(\widetilde{R}[a,b], \widetilde{d_1})$. A similar consideration applies to the L^2-distance to get a metric $\widetilde{d_2}$.

A **norm** $\|\cdot\|$ is a function on a real vector space X to $[0,\infty)$ satisfying the following three conditions, for all $x, y \in X$ and $\alpha \in \mathbb{R}$,

N1. $\|x\| \geq 0$ and "=" 0 if and only if $x = 0$

N2. $\|\alpha x\| = |\alpha|\ \|x\|$, and

N3. $\|x+y\| \leq \|x\| + \|y\|$.

The pair $(X, \|\cdot\|)$ is called a **normed space**. There is always a metric associated to a norm. Indeed, letting

$$d(x,y) = \|x-y\|,$$

it is readily checked that d defines a metric on X. This metric is called the **metric induced** by the norm. In all the five examples above the metrics are induced respectively from norms. I leave it to you to write down the corresponding norms. Normed spaces will be studied in MATH4010 Functional Analysis. In the following we give two examples of metrics defined on a set without the structure of a vector space. Hence they cannot be metrics induced by norms.

Example 2.6. Let X be a non-empty set. For $x, y \in X$, define

$$d(x,y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

The metric d is called the **discrete metric** on X. The metric ball $B_r(x)$ consists of x itself for all $r \in (0,1)$.

Example 2.7. Let H be the collection of all strings of words in n digits. For two strings of words in H, $a = a_1 \cdots a_n$, $b = b_1 \cdots b_n$, $a_j, b_j \in \{0,1,2,\ldots,9\}$. Define

$$d_H(a,b) = \text{the number of digits at which } a_j \text{ is not equal to } b_j.$$

By using a simple induction argument one can show that (H, d_H) forms a metric space. Indeed, the case $n = 1$ is straightforward. Let us assume it holds for n-strings and show it for $(n+1)$-strings. Let $a = a_1 \cdots a_n a_{n+1}, b = b_1 \cdots b_n b_{n+1}, c = c_1 \cdots c_n c_{n+1}, a' =$

$a_1 \cdots a_n, b' = b_1 \cdots b_n$, and $c' = c_1 \cdots c_n$. Consider the case that $a_{n+1} = b_{n+1} = c_{n+1}$. We have $d_H(a,b) = d_H(a',b') \leq d_H(a',c') + d_H(c',b') = d_H(a,c) + d_H(c,b)$ by induction hypothesis. When a_{n+1} is not equal to one of b_{n+1}, c_{n+1}, say, $a_{n+1} \neq b_{n+1}$, we have $d_H(a,b) = d_H(a',b')+1 \leq d_H(a',c')+d_H(c',b')+1$. If $b_{n+1} = c_{n+1}$, $d_H(a,c) = d_H(a',c')+1$. Therefore, $d_H(a',c') + d_H(c',b') + 1 \leq d_H(a,c) + d_H(c,b)$. If $b_{n+1} \neq c_{n+1}$, $d_H(c',b') = d_H(c,b) - 1$ and the same inequality holds. Finally, if $a_{n+1} \neq c_{n+1}$ and $a_{n+1} = b_{n+1}$, $d_H(a,b) = d_H(a',b') \leq d_H(a',c') + d_H(c',b') \leq d_H(a,c) - 1 + d_H(c,b) - 1 \leq d_H(a,c) + d_H(c,b)$. The metric d_H is called the **Hamming distance**. It measures the error in a string during transmission.

Let Y be a non-empty subset of (X,d). Then $(Y, d|_{Y\times Y})$ is again a metric space. It is called a **metric subspace** of (X,d). The notation $d|_{Y\times Y}$ is usually written as d for simplicity. Every non-empty subset of a metric space forms a metric space under the restriction of the metric. In the following we usually call a metric subspace a subspace for simplicity. Note that a metric subspace of a normed space needs not be a normed space. It is so only if the subset is also a vector subspace.

Recall that convergence of sequences of real numbers and uniform convergence of sequences of functions are main themes in MATH2050/60 and sequences of vectors were considered in MATH2010/20. With a metric d on a set X, it makes sense to talk about limits of sequences in a metric space. Indeed, a sequence in (X,d) is a map φ from $\mathbb{N}$ to (X,d) and usually we write it in the form $\{x_n\}$ where $\varphi(n) = x_n$. We call $\{x_n\}$ **converges to** x if $\lim_{n\to\infty} d(x_n,x) = 0$, that's, for every $\varepsilon > 0$, there exists n_0 such that $d(x_n,x) < \varepsilon$, for all $n \geq n_0$. When this happens, we write or $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ in X.

Convergence of sequences in $(\mathbb{R}^n, d_2)$ reduces to the old definition we encountered before. From now on, we implicitly refer to the Euclidean metric when convergence of sequences in $\mathbb{R}^n$ is considered. For sequences of functions in $(C[a,b], d_\infty)$, it is simply the uniform convergence of sequences of functions in $C[a,b]$.

As there could be more than one metrics defined on the same set, it is natural to make a comparison among these metrics. Let d and ρ be two metrics defined on X. We call ρ is **stronger** than d, or d is **weaker** than ρ, if there exists a positive constant C such that $d(x,y) \leq C\rho(x,y)$ for all $x,y \in X$. They are **equivalent** if d is stronger and weaker than ρ simultaneously, in other words,

$$d(x,y) \leq C_1\rho(x,y) \leq C_2 d(x,y), \quad \forall x,y \in X,$$

for some positive C_1 and C_2. When ρ is stronger than d, a sequence converging in ρ is also convergent in d. When d and ρ are equivalent, a sequence is convergent in d if and only if it is so in ρ.

Take d_1, d_2 and d_∞ on $\mathbb{R}^n$ as an example. It is elementary to show that for all $x,y \in \mathbb{R}^n$,

$$d_2(x,y) \leq n^{1/2} d_\infty(x,y) \leq n^{1/2} d_2(x,y),$$

and

$$d_1(x, y) \leq n d_\infty(x, y) \leq n d_1(x, y),$$

hence d_1, d_2 and d_∞ are all equivalent. The convergence of a sequence in one metric implies its convergence in other two metrics.

It is a basic result in functional analysis that every two induced metrics in a finite dimensional normed space are equivalent. Consequently, examples of inequivalent metrics can only be found when the underlying space is of infinite dimensional.

Let us display two inequivalent metrics on $C[a, b]$. For this purpose it suffices to consider d_1 and d_∞. On one hand, clearly we have

$$d_1(f, g) \leq (b - a) d_\infty(f, g), \quad \forall f, g \in C[a, b],$$

so d_∞ is stronger than d_1. But the converse is not true. Consider the sequence given by (taking $[a, b] = [0, 1]$ for simplicity)

$$f_n(x) = \begin{cases} -n^3 x + n, & x \in [0, 1/n^2], \\ 0, & x \in (1/n^2, 1]. \end{cases}$$

We have $d_1(f_n, 0) \to 0$ but $d_\infty(f_n, 0) \to \infty$ as $n \to \infty$. Were $d_\infty(f_n, 0) \leq C d_1(f_n, 0)$ true for some positive constant C, $d_1(f_n, 0)$ must tend to ∞ as well. Now it tends to 0, so d_1 cannot be stronger than d_2 and these two metrics are not equivalent.

Now we define continuity in a metric space. Recalling that for a real-valued function defined on some set E in $\mathbb{R}$, there are two equivalent ways to define the continuity of the function at a point. We could use either the behavior of sequences or the ε-δ formulation. Specifically, the function f is continuous at $x \in E$ if for every sequence $\{x_n\} \subset E$ satisfying $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} f(x_n) = f(x)$. Equivalently, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ whenever $y \in E, |y - x| < \delta$. Both definition can be formulated on a metric space. Let (X, d) and (Y, ρ) be two metric spaces and $f : (X, d) \to (Y, \rho)$. Let $x \in X$. We call f is **continuous at** x if $f(x_n) \to f(x)$ in (Y, ρ) whenever $x_n \to x$ in (X, d). It is **continuous** on a set $E \subset X$ if it is continuous at every point of E.

Proposition 2.1. *Let f be a mapping from (X, d) to (Y, ρ) and $x_0 \in X$. Then f is continuous at x_0 if and only if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ for all $x, d(x, x_0) < \delta$.*

Proof. $\Leftarrow$) Let ε be given and δ is chosen accordingly. For any $\{x_n\} \to x_0$, given $\delta > 0$, there exists some n_0 such that $d(x_n, x_0) < \delta \ \forall n \geq n_0$. It follows that $\rho(f(x_n), f(x_0)) < \varepsilon$ for all $n \geq n_0$, so f is continuous at x_0.

$\Rightarrow$) Suppose that the implication is not valid. There exist some $\varepsilon_0 > 0$ and $\{x_k\} \in X$ satisfying $d(f(x_k), f(x_0)) \geq \varepsilon_0$ and $d(x_k, x_0) < 1/k$. However, the second condition tells us that $\{x_k\} \to x_0$, so by the continuity at x_0 one should have $d(f(x_k), f(x_0)) \to 0$, contradiction holds. $\square$

We will shortly use open/closed sets to describe continuity in a metric space.

As usual, continuity of functions is closed under compositions of functions.

Proposition 2.2. *Let $f : (X, d) \to (Y, \rho)$ and $g : (Y, \rho) \to (Z, m)$ be given.*

(a) If f is continuous at x and g is continuous at $f(x)$, then $g \circ f : (X, d) \to (Z, m)$ is continuous at x.

(b) If f is continuous in X and g is continuous in Y, then $g \circ f$ is continuous in X.

Proof. It suffices to prove (a). Let $x_n \to x$. Then $f(x_n) \to f(x)$ as f is continuous at x. Then $(g \circ f)(x_n) = g(f(x_n)) \to g(f(x)) = (g \circ f)(x)$ as g is continuous at $f(x)$. □

2.2 Open and Closed Sets

The existence of a metric on a set enables us to talk about convergence of a sequence and continuity of a map. It turns that in order to define continuity it requires structure less stringent then a metric structure. It suffices the set is endowed with a topological structure. In a word, a metric induces a topological structure on the set but not every topological structure comes from a metric. In a topological space, continuity can no longer be defined via the convergence of sequences. Instead one uses the notion of open and closed sets in the space. As a warm up for topology we discuss how to use the language of open/closed sets to describe the convergence of sequences and the continuity of functions in this section.

Let (X, d) be a metric space. A set $G \subset X$ is called an **open set** if for each $x \in G$, there exists some ρ such that $B_\rho(x) \subset G$. The number ρ may vary depending on x. We also define the empty set ϕ to be an open set.

Proposition 2.3. *Let (X, d) be a metric space. We have*

(a) X and ϕ are open sets.

(b) If $\bigcup_{\alpha \in \mathcal{A}} G_\alpha$ is an open set provided that all G_α, $\alpha \in \mathcal{A}$, are open where $\mathcal{A}$ is an arbitrary index set.

(c) If $G_1, \dots, G_N$ are open sets, then $\bigcap_{j=1}^N G_j$ is an open set.

Note the union in (b) of this proposition is over an arbitrary collection of sets while the intersection in (c) is a finite one.

Proof. (a) Obvious.

(b) Let $x \in \bigcup_{\alpha\in\mathcal{A}} G_\alpha$. There exists some α_1 such that $x \in G_{\alpha_1}$. As G_{α_1} is open, there is some $B_\rho(x) \subset G_{\alpha_1}$. But then $B_\rho(x) \subset \bigcup_{\alpha\in\mathcal{A}} G_\alpha$, so $\bigcup_{\alpha\in\mathcal{A}} G_\alpha$ is open.

(c) Let $x \in \bigcap_{j=1}^N G_j$. For each j, there exists $B_{\rho_j}(x) \subset G_j$. Let $\rho = \min\{\rho_1, \dots, \rho_N\}$. Then $B_\rho(x) \subset \bigcap_{j=1}^N G_j$, so $\bigcap_{j=1}^N G_j$ is open. □

The complement of an open set is called a **closed set**. Taking the complement of Proposition 2.2, we have

Proposition 2.4. *Let (X, d) be a metric space. We have*

(a) X and ϕ are closed sets.

(b) If F_α, $\alpha \in \mathcal{A}$, are closed sets, then $\bigcap_{\alpha\in\mathcal{A}} F_\alpha$ is a closed set.

(c) If $F_1, \dots, F_N$ are closed sets, then $\bigcup_{j=1}^N F_j$ is a closed set.

Note that X and ϕ are both open and closed.

Example 2.8. Every ball in a metric space is an open set. Let $B_r(x)$ be a ball and $y \in B_r(x)$. We claim that $B_\rho(y) \subset B_r(x)$ where $\rho = r - d(y, x) > 0$. For, if $z \in B_\rho(y)$,

$$\begin{aligned} d(z,x) &\le d(z,y) + d(y,x) \\ &< \rho + d(y,x) \\ &= r\ , \end{aligned}$$

by the triangle inequality, so $z \in B_r(x)$ and $B_\rho(y) \subset B_r(x)$ holds. Next, the set $E = \{y \in X : d(y, x) > r\}$ for fixed x and $r \ge 0$ is an open set. For, let $y \in E$, $d(y, x) > r$. We claim $B_\rho(y) \subset E$, $\rho = d(y, x) - r > 0$,. For, letting $z \in B_\rho(y)$,

$$\begin{aligned} d(z,x) &\ge d(y,x) - d(y,z) \\ &> d(y,x) - \rho \\ &= r, \end{aligned}$$

shows that $B_\rho(y) \subset E$, hence E is open. Finally, consider $F = \{x \in X : d(x, z) = r > 0\}$ where z and r are fixed. Observing that F is the complement of the two open sets $B_r(z)$ and $\{x \in X : d(x, z) > r\}$, we conclude that F is a closed set.

Example 2.9. In the real line every open interval (a, b), $-\infty \le a \le b \le \infty$, is an open set. Other intervals such as $[a, b)$, $[a, b]$, $(a, b]$, $a, b \in \mathbb{R}$, are not open. It can be shown that every open set G in $\mathbb{R}$ can be written as a disjoint union of open intervals. Letting $(a_n, b_n) = (-1/n, 1/n)$,

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not open. It shows that Proposition 2.2(c) does not hold when the intersection is over infinite many sets. On the other hand, $[a, b]$ is a closed set since $\{a\} = \mathbb{R}\setminus(-\infty, a)\cup(a, \infty)$, a single point is always a closed set. Finally, some sets we encounter often are neither open nor closed. Take the set of all rational numbers as example, as every open interval containing a rational number also contains an irrational number, we see that $\mathbb{Q}$ is not open. The same reasoning shows that the set of all irrational numbers is not open, hence $\mathbb{Q}$ is also not a closed set.

Example 2.10. When we studied multiple integrals in MATH2020, we encountered many domains or regions as the domain of integration. These domains are open sets in $\mathbb{R}^n$. For instance, consider the set $G = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1^2/4 + x_2^2/9 < 1\}$ which is the set of all points lying inside an ellipse. We claim that it is an open set. Each point $(y_1, y_2) \in G$ satisfies the inequality

$$\frac{y_1^2}{4} + \frac{y_2^2}{9} < 1.$$

Since the function $(x_1, x_2) \mapsto x_1^2/4 + x_2^2/9 - 1$ is continuous, there exists some $\varepsilon > 0$ such that

$$\frac{z_1^2}{4} + \frac{z_2^2}{9} < 1,$$

for all $(z_1, z_2), \ d((z_1, z_2), (y_1, y_2)) < \varepsilon$. In other words, the ball $B_\varepsilon((y_1, y_2))$ is contained in G, so G is open. Similarly, one can show that the outside of the ellipse, denoted by H, is open (using the fact $x_1^2/4 + x_2^2/9 > 1$ in H) and the set composed of all points lying on the ellipse, denoted by S, is closed. Finally, the set $G \cup S$ is closed as its complement H is open. In general, most domains in $\mathbb{R}^2$ in advanced calculus consist of points bounded by one or several continuous curves. They are all open sets like G. All points lying outside of the boundary curves form an open set and those lying on the curves form a closed set. The points sitting inside and on the curves form an closed set. The situation extends to higher dimensional domains whose boundary are given by finitely many pieces of continuous hypersurfaces.

Example 2.11. Consider the set $E = \{f \in C[a, b] : \ f(x) > 0, \ \forall x \in [a, b]\}$ in $C[a, b]$. We claim that it is open. For $f \in E$, it is positive everywhere on the closed, bounded interval $[a, b]$, hence it attains its minimum at some x_0. It follows that $f(x) \geq m \equiv f(x_0) > 0$. Letting $r = m/2$, for $g \in B_r(f), \ d_\infty(g, f) < r = m/2$ implies

$$\begin{aligned} g(x) &\geq f(x) - |g(x) - f(x)| \\ &> m - \frac{m}{2} \\ &= \frac{m}{2} > 0, \end{aligned}$$

for all $x \in [a,b]$, hence $g \in E$ which implies $B_r(f) \subset E$, E is open. Likewise, sets like $\{f : \ f(x) > \alpha, \ \forall x\}, \{f : \ f(x) < \alpha, \ \forall x\}$ where α is a fixed number. On the other hand, by taking complements of these open sets, we see that the sets $\{f : \ f(x) \geq \alpha, \ \forall x\}, \{f : f(x) \leq \alpha, \ \forall x\}$ are closed.

Example 2.12. Consider the extreme case where the space X is endowed with the discrete metric. We claim that every set is open and closed. Clearly, it suffices to show that every singleton set $\{x\}$ is open. But, this is obvious because the ball $B_{1/2}(x) = \{x\}$ belongs to $\{x\}$. It is also true that $B_r(x) = X$ once $r > 1$.

We now use open sets to describe the convergence of sequences.

Proposition 2.5. *Let (X,d) be a metric space. A sequence $\{x_n\}$ converges to x if and only if for each open G containing x, there exists n_0 such that $x_n \in G$ for all $n \geq n_0$.*

Proof. Let G be an open set containing x. According to the definition of an open set, we can find $B_\varepsilon(x) \subset G$. It follows that there exists n_0 such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$, i.e., $x_n \in B_\varepsilon(x) \subset G$ for all $n \geq n_0$. Conversely, taking $G = B_\varepsilon(x)$, we see that $x_n \to x$. □

From this proposition we deduce the following result which explains better the terminology of a closed set.

Proposition 2.6. *The set A is a closed set in (X,d) if and only if whenever $\{x_n\} \subset A$ and $x_n \to x$ as $n \to \infty$ implies that x belongs to A.*

Proof. $\Rightarrow$). Assume on the contrary that x does not belong to A. As $X \setminus A$ is an open set, by Proposition 2.4 we can find a ball $B_\varepsilon(x) \subset X \setminus A$. However, as $x_n \to x$, there exists some n_0 such that $x_n \in B_\varepsilon(x)$ for all $n \geq n_0$, contradicting the fact that $x_n \in A$.

$\Leftarrow$). If $X \setminus A$ is not open, say, we could find a point $x \in X \setminus A$ such that $B_{1/n}(x) \bigcap A \neq \phi$ for all n. Pick $x_n \in B_{1/n}(x) \bigcap A$ to form a sequence $\{x_n\}$. Clearly $\{x_n\}$ converges to x. By assumption, $x \in A$, contradiction holds. Hence $X \setminus A$ must be open.

□

Now we use open sets to describe the continuity of functions.

Proposition 2.7. *Let $f : (X,d) \to (Y,\rho)$.*

(a) f is continuous at x if and only if for every open set G containing $f(x)$, $f^{-1}(G)$ contains $B_\varepsilon(x)$ for some $\varepsilon > 0$.

(b) f is continuous in X if and only if for every open G in Y, $f^{-1}(G)$ is an open set in X.

These statements are still valid when "open" is replaced by "closed".

Proof. We consider (a) and (b) comes from (a) easily.

$\Rightarrow$). Suppose there exists some open G such that $f^{-1}(G)$ does not contain $B_{1/n}(x)$ for all $n \geq 1$. Pick $x_n \in B_{1/n}(x)$, $x_n \notin f^{-1}(G)$. Then $x_n \to x$ but $f(x_n)$ does not converge to x, contradicting the continuity of f.

$\Leftarrow$). Let $\{x_n\} \to x$ in X. Given any open set G containing $f(x)$, we can find $B_r(x) \subset f^{-1}(G)$. Thus, there exists n_0 such that $x_n \in B_r(x)$ for all $n \geq n_0$. It follows that $f(x_n) \in G$ for all $n \geq n_0$. By Proposition 2.4, f is continuous at x.

□

Let Y be a subspace of (X, d). We describe the open sets in Y. First of all, the metric ball in (Y, d) is given by $B'_r(x) = \{y \in Y : d(y, x) < r\}$ which is equal to $B_r(x) \cap Y$. For an open set E in Y, for each $x \in E$ there exists some $B'_{\rho_x}(x)$, such that $B'_{\rho_x}(x) \subset E$. Therefore,

$$E = \bigcup B'_{\rho_x}(x) = \bigcup (B_{\rho_x}(x) \cap Y) = \left(\bigcup B_{\rho_x}(x)\right) \cap Y.$$

We conclude

Proposition 2.8. *Let Y be a subspace of (X, d). A set E in Y is open in Y if and only if there exists an open set G in (X, d) satisfying $E = G \cap Y$. It is closed if and only if there exists a closed set F in (X, d) satisfying $E = F \cap Y$.*

Example 2.13. Let $[0, 1]$ be the subspace of $\mathbb{R}$ under the Euclidean metric. The set $[0, 1/2)$ is not open in $\mathbb{R}$ as every open interval of the form $(a, b), a < 0 < b$, is not contained in $[0, 1/2)$, so 0 is not an interior point of $[0, 1)$. However, it is relatively open in $[0, 1)$ because when regarded as a subset of $[0, 1)$, the set $[0, a), 1/2 > a > 0$, is an open set (relative in $[0, 1)$) contained in $[0, 1/2)$. For, by the proposition above, $[0, a) = (-1, a) \cap [0, 1)$ is relatively open.

We describe some further useful notions associated to sets in a metric space.

Let E be a set in (X, d). A point x is called a **boundary point** of E if $G \cap E$ and $G \setminus E$ are non-empty for every open set G containing x. Of course, it suffices to take G of the form $B_\varepsilon(x)$ for all sufficiently small ε or $\varepsilon = 1/n,\ n \geq 1$. We denote the boundary of E by ∂E. The **closure** of E, denoted by $\overline{E}$, is defined to be $E \cup \partial E$. Clearly $\partial E = \partial(X \setminus E)$. The boundary of the ball $B_r(x)$ in $\mathbb{R}^n$ is the sphere $S_r(x) = \{y \in \mathbb{R}^n : \ d_2(y, x) = r\}$. Hence, the closed ball $\overline{B_r(x)}$ is given by $B_r(x) \bigcup S_r(x)$, which is precisely the closure of $B_r(x)$.

Example 2.14. Let $E = [0,1) \times [0,1)$. It is easy to see that $\partial E = [0,1] \times \{0,1\} \cup \{0,1\} \times [0,1]$. Thus some points in ∂E belong to E and some do not. The closure of E, $\overline{E}$, is equal to $[0,1] \times [0,1]$.

It can be seen from definition that the boundary of the empty set is the empty set and the boundary of a set is always a closed set. For, let $\{x_n\}$ be a sequence in ∂E converging to some x. For any ball $B_r(x)$, we can find some x_n in it, so the ball $B_\rho(x_n)$, $\rho = r - d(x_n, x) > 0$, is contained in $B_r(x)$. As $x_n \in \partial E$, $B_\rho(x_n)$ has non-empty intersection with E and $X \setminus E$, so does $B_r(x)$ and $x \in \partial E$ too. The following proposition characterizes the closure of a set as the smallest closed set containing this set.

Proposition 2.9. *Let E be a set in (X, d). We have*

$$\overline{E} = \bigcap \{C : \ C \text{ is a closed set containing } E\}.$$

Proof. We first claim that $\overline{E}$ is a closed set. We will do this by showing $X \setminus \overline{E}$ is open. Indeed, for x lying outside $\overline{E}$, x does not belong to E and there exists an open ball $B_\rho(x)$ disjoint from E. Thus, $\overline{E}$ is disjoint from $B_{\rho/2}(x)$ and so $X \setminus \overline{E}$ is open. We conclude that $\overline{E}$ is closed. Next we claim that $\overline{E}$ is contained in any closed set C containing E. It suffices to show that $\partial E \subset C$. Indeed, if $x \in \partial E$, every ball $B_{1/n}(x)$ would have non-empty intersection with E. By picking a point x_n from $B_{1/n}(x) \cap E$, we obtain a sequence $\{x_n\}$ in E converging to x as $n \to \infty$. As C is closed, x belongs to C by Proposition 2.5. $\square$

A point x is called an **interior point** of E if there exists an open set G containing x such that $G \subset E$. It can be shown that all interior points of E form an open set call the **interior** of E, denoted by E^o. It is not hard to see that $E^0 = E \setminus \partial E$. The interior of a set is related to its closure by the following relation: $E^o = X \setminus \left(\overline{X \setminus E}\right)$. Using this relation, one can show that the interior of a set is the largest open set sitting inside E. More precisely, $G \subset E^0$ whenever G is an open set in E.

Example 2.15. In Example 2.10 we consider domains in $\mathbb{R}^2$ bounded by several pieces of continuous curves. Let D be such a domain and the curves bounding it be S. It is routine to verify that $\partial D = S$, that is, the set of all boundary points of D is precisely the S and the closure of D, $\overline{D}$, is $D \cup S$. The interior of $\overline{D}$ is D.

2.3 Compactness

Recall that Bolzano-Weierstrass theorem asserts that every sequence in a closed bounded interval has a convergent subsequence in this interval. The result still holds for all closed, bounded sets in $\mathbb{R}^n$. In general, a set $E \subset (X, d)$ is **compact** if every sequence has a convergent subsequence with limit in E. This property is also called **sequentially compact** to stress that the behavior of sequences is involved in the definition. The space (X, d) is called a **compact space** is X is a compact set itself. According to this definition, every interval of the form $[a, b]$ is compact in $\mathbb{R}$ and sets like $[a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n]$ and $\overline{B_r(x)}$ are compact in $\mathbb{R}^n$ under the Euclidean metric. In a general metric space, the notion of a bounded set makes perfect sense. Indeed, a set A is called a **bounded set** if there exists some ball $B_r(x)$ for some $x \in X$ and $r > 0$ such that $A \subset B_r(x)$. Now we investigate the relation between a compact set and a closed bounded set. First of all, we have

Proposition 2.10. *Every compact set in a metric space is closed and bounded.*

Proof. Let K be a compact set. To show that it is closed, let $\{x_n\} \subset K$ and $x_n \to x$. We need to show that $x \in K$. As K is compact, there exists a subsequence $\{x_{n_j}\} \subset K$ converging to some z in K. By the uniqueness of limit, we have $x = z \in K$, so $x \in K$ and K is closed. On the other hand, if K is unbounded, that is, for any fixed point x_0, K is not contained in the balls $B_n(x_0)$ for all n. Picking $x_n \in K \setminus B_n(x_0)$, we obtain a sequence $\{x_n\}$ satisfying $d(x_n, x_0) \to \infty$ as $n \to \infty$. By the compactness of K, there is a subsequence $\{x_{n_j}\}$ converging to some z in K. By the triangle inequality,

$$
\begin{aligned}
\infty > d(z, x_0) &= \lim_{j\to\infty} d(x_{n_j}, z) + d(z, x_0) \\
&\geq \lim_{j\to\infty} d(x_{n_j}, x_0) \to \infty,
\end{aligned}
$$

as $j \to \infty$, contradiction holds. Hence K must be bounded. □

As a consequence of Bolzano-Weierstrass theorem every sequence in a bounded and closed set in $\mathbb{R}^n$ contains a convergent subsequence. Thus a set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Proposition 2.10 tells that every compact set is in general closed and bounded, but the converse is not always true. To describe an example we need to go beyond $\mathbb{R}^n$ where we can be free of the binding of Bolzano-Weierstrass theorem. Consider the set $\mathcal{S} = \{f \in C[0, 1] : 0 \leq f(x) \leq 1\}$. Clearly it is closed and bounded in $C[0, 1]$. We claim that it is not compact. For, consider the sequence $\{f_n\}$ in $(C[0, 1], d_\infty)$ given by

$$
f_n(x) = \begin{cases} nx, & x \in [0, \frac{1}{n}] \\ 1, & x \in [\frac{1}{n}, 1]. \end{cases}
$$

$\{f_n(x)\}$ converges pointwisely to the function $f(x) = 1$, $x \in (0, 1]$ and $f(0) = 0$ which is discontinuous at $x = 0$, that is, f does not belong to $C[0, 1]$. If $\{f_n\}$ has a convergent subsequences, then it must converge uniformly to f. But this is impossible because the uniform limit of a sequence of continuous functions must be continuous. Hence $\mathcal{S}$ cannot be compact. In fact, a remarkable theorem in functional analysis asserts that the closed unit ball in a normed space is compact if and only if the normed space if and only if the normed space is of finite dimension.

Since convergence of sequences can be completely described in terms of open/closed sets, it is natural to attempt to describe the compactness of a set in terms of these new notions. The answer to this challenging question is a little strange at first sight. We introduce some terminologies. First of all, an **open cover** of a subset E in a metric space (X, d) is a collection of open sets $\{G_\alpha\}, \alpha \in \mathcal{A}$, satisfying $E \subset \bigcup_{\alpha\in\mathcal{A}} G_\alpha$. A set $E \subset X$ satisfies the **finite cover property** if whenever $\{G_\alpha\}, \alpha \in \mathcal{A}$, is an open cover of E, there exist a subcollection consisting of finitely many $G_{\alpha_1}, \ldots, G_{\alpha_N}$ such that $E \subset \bigcup_{j=1}^N G_{\alpha_j}$. ("Every open cover has a finite subcover.") A set E satisfies the **finite intersection property** if whenever $\{F_\alpha\}, \alpha \in \mathcal{A}$, are relatively closed sets in E satisfying $\bigcap_{j=1}^N F_{\alpha_j} \neq \phi$ for any finite subcollection F_{α_j}, $\bigcap_{\alpha\in\mathcal{A}} F_\alpha \neq \phi$. Here a set $F \subset E$ is relatively closed means F is closed in the subspace E. We know that it implies $F = A \cap E$ for some closed set A. Therefore, when E is closed, a relatively closed subset is also closed.

Proposition 2.11. *A closed set has the finite cover property if and only if it has the finite intersection property.*

Proof. Let E be a non-empty closed set in (X, d).

$\Rightarrow$) Suppose $\{F_\alpha\}$, F_α closed sets contained in E, satisfies $\bigcap_{j=1}^N F_{\alpha_j} \neq \phi$ for any finite subcollection but $\bigcap_{\alpha\in\mathcal{A}} F_\alpha = \phi$. As E is closed, each F_α is closed in X, and

$$E = E \setminus \bigcap_{\alpha\in\mathcal{A}} F_\alpha = \bigcup_{\alpha\in\mathcal{A}} (E \cap F'_\alpha) \subset \bigcup_{\alpha\in\mathcal{A}} F'_\alpha.$$

By the finite covering property we can find $\alpha_1, \ldots, \alpha_N$ such that $E \subset \bigcup_{j=1}^N F'_{\alpha_j}$, but then $\phi = E \setminus E \supset E \setminus \bigcup_1^N F'_{\alpha_j} = \bigcap_{j=1}^N F_{\alpha_j}$, contradiction holds.

$\Leftarrow$) If $E \subset \bigcup G_{\alpha\in\mathcal{A}}$ but $E \subsetneq \bigcup_{j=1}^N G_{\alpha_j}$ for any finite subcollection of $\mathcal{A}$, then

$$\phi \neq E \setminus \bigcup_{j=1}^N G_{\alpha_j} = \bigcap_{j=1}^N \left(E \setminus G_{\alpha_j}\right)$$

which implies $\bigcap_{\alpha\in\mathcal{A}} \left(E \setminus G_\alpha\right) \neq \phi$ by the finite intersection property. Note that each $E \setminus G_{\alpha_j}$ is closed. Using $E \bigcap \left(\bigcup_{\alpha\in\mathcal{A}} G_\alpha\right)' = \bigcap_{\alpha\in\mathcal{A}} \left(E \setminus G_\alpha\right)$, we have $E \subsetneq \bigcup_{\alpha\in\mathcal{A}} G_\alpha$, contradicting our assumption.

□

Proposition 2.12. *Let E be compact in a metric space. For each $\alpha > 0$, there exist finitely many balls $B_\alpha(x_1), \ldots, B_\alpha(x_N)$ such that $E \subset \bigcup_{j=1}^N B_\alpha(x_j)$ where $x_j, 1 \le j \le N$, are in E.*

Proof. Pick $B_\alpha(x_1)$ for some $x_1 \in E$. Suppose $E \setminus B_\alpha(x_1) \neq \phi$. We can find $x_2 \notin B_\alpha(x_1)$ so that $d(x_2, x_1) \ge \alpha$. Suppose $E \setminus \left(B_\alpha(x_1) \bigcup B_\alpha(x_2)\right)$ is non-empty. We can find $x_3 \notin$

$B_\alpha(x_1) \bigcup B_\alpha(x_2)$ so that $d(x_j, x_3) \geq \alpha,\ j = 1, 2$. Keeping this procedure, we obtain a sequence $\{x_n\}$ in E such that

$$E \setminus \bigcup_{j=1}^{n} B_\alpha(x_j) \neq \phi \quad \text{and} \quad d(x_j, x_n) \geq \alpha,\ j = 1, 2, \ldots, n-1.$$

By the compactness of E, there exists $\{x_{n_j}\}$ and $x \in E$ such that $x_{n_j} \to x$ as $j \to \infty$. But then $d(x_{n_j}, x_{n_k}) < d(x_{n_j}, x) + d(x_{n_k}, x) \to 0$, contradicting $d(x_j, x_n) \geq \alpha$ for all $j < n$. Hence one must have $E \setminus \bigcup_{j=1}^{N} B_\alpha(x_j) = \phi$ for some finite N.

□

Sometimes the following terminology is convenient. A set E is called **totally bounded** if for each $\varepsilon > 0$, there exist $x_1, \cdots, x_n \in X$ such that $E \subset \cup_{k=1}^{n} B_\varepsilon(x_k)$. Proposition 2.12 simply states that every compact set is totally bounded. We will use this property of a compact set again in the next chapter.

Theorem 2.13. *Let E be a closed set in (X, d). The followings are equivalent:*

(a) E is compact;

(b) E satisfies the finite cover property; and

(c) E satisfies the finite intersection property.

Proof. (a)⇒ (b). Let $\{G_\alpha\}$ be an open cover of E without finite subcover and we will draw a contradiction. By Proposition 2.11, for each $k \geq 1$, there are finitely many balls of radius $1/k$ covering E. We can find a set $B_{1/k} \cap E$ (suppress the irrelevant center) which cannot be covered by finitely many members in $\{G_\alpha\}$. Pick $x_k \in B_{1/k} \cap E$ to form a sequence. By the compactness of E, we can extract a subsequence $\{x_{k_j}\}$ such that $x_{k_j} \to x$ for some $x \in E$. Since $\{G_\alpha\}$ covers E, there must be some G_β that contains x. As G_β is open and the radius of B_{1/k_j} tends to 0, we deduce that, for all sufficiently large k_j, $B_{1/k_j} \cap E$ is contained in G_β. In other words, G_β forms a single subcover of $B_{1/k} \cap E$, contradicting our choice of $B_{1/k_j} \cap E$. Hence (b) must be valid.

(b)⇔ (c). See Proposition 2.11.

(c)⇒ (a). Let $\{x_n\}$ be a sequence in E. Without loss of generality we may assume that it contains infinitely many distinct points, otherwise the conclusion is obvious. The balls $B_1(x_n)$ form an open cover of the set $\overline{\{x_n\}}$, hence it has a finite subcover. (Note that a closed subset of a set satisfying the finite cover property again satisfies the finite cover property.) Since there are infinitely many distinct points in this sequence, we can choose one of the these balls, denoted by B_1, which contains infinitely many points in E. Next we cover $\overline{\{x_n\}}$ by balls $B_{1/2}(x_n)$. Among a finite subcover, choose one of the balls $B_{1/2}$ which contains infinitely many distinct points from B_1. Keeping doing this, we obtain a sequence of balls $\{B_{1/k}\}, k \geq 1$, such that each $B_1 \cap \cdots \cap B_{1/k}$ contains infinitely many distinct points in E. We pick a subsequence $\{x_{n_k}\}$ from $B_1 \cap \cdots \cap B_{1/k} \cap E$. By the finite intersection property, $\bigcap_k \left(\overline{B}_{1/k} \cap E\right)$ is nonempty and in fact consists of one point $z \in E$. It is clear that $d(x_{n_k}, z) \leq 2/k \to 0$ as $k \to \infty$. We have succeeded in producing a convergent subsequence in the closed set E. Hence E is compact. $\square$

We finally note

Proposition 2.14. *Let E be a compact set in (X, d) and $F : (X, d) \to (Y, \rho)$ be continuous. Then $f(E)$ is a compact set in (Y, ρ).*

Proof. Let $\{y_n\}$ be a sequence in $f(E)$ and let $\{x_n\}$ be in E satisfying $f(x_n) = y_n$ for all n. By the compactness of E, there exist some $\{x_{n_j}\}$ and x in E such that $x_{n_j} \to x$ as $j \to \infty$. By the continuity of f, we have $y_{n_j} = f(x_{n_j}) \to f(x)$ in $f(E)$. Hence $f(E)$ is compact. $\square$

Can you prove this property by using the finite cover property of compact sets?

There are several fundamental theorems which hold for continuous functions defined on a closed, bounded set in the Euclidean space. They include a continuous function on such a set is uniformly continuous and attains its minimum and maximum. Although they may no longer hold on arbitrary closed, bounded sets in a general metric space, they continue to hold when the sets are strengthened to compact ones. The proofs are very much like in the finite dimensional case. I leave them as exercises.

2.4 Completeness

In $\mathbb{R}^n$ a basic property is that every Cauchy sequence converges. This property is called the completeness of the Euclidean space. The notion of a Cauchy sequence is well-defined in a metric space. Indeed, a sequence $\{x_n\}$ in (X, d) is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists some n_0 such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq n_0$. A metric space (X, d) is **complete** if every Cauchy sequence converges. A subset E is **complete** if $(E, d\big|_{E \times E})$ is complete.

Example 2.16. The interval $[a, b]$ is a complete space. For, if $\{x_n\}$ is a Cauchy sequence in $[a, b]$, it is also a Cauchy sequence in $\mathbb{R}$. By the completeness of the real line, $\{x_n\}$ converges to some x. Since $[a, b]$ is closed, x must belong to $[a, b]$, so $[a, b]$ is complete. In contrast, the set $[a, b)$, $b \in \mathbb{R}$, is not complete. For, simply observe that the sequence $\{b - 1/k\}, k \geq k_0$ for some large k_0, is a Cauchy sequence in $[a, b)$ and yet it does not have a limit in $[a, b)$ (the limit is b, which lies outside $[a, b)$).

Example 2.17. In MATH2060 we learned that every Cauchy sequence in $C[a, b]$ with respect to the sup-norm implies that it converges uniformly, so the limit is again continuous and $C[a, b]$ is a complete space. The subset $E = \{f : f(x) \geq 0, \ \forall x\}$ is also complete. Let $\{f_n\}$ be a Cauchy sequence in E, it is also a Cauchy sequence in $C[a, b]$ and hence there exists some $f \in C[a, b]$ such that $\{f_n\}$ converges to f uniformly. As uniform convergence implies pointwise convergence, $f(x) = \lim_{n\to\infty} f_n(x) \geq 0$, so f belongs to E and E is complete. Next, let $P[a, b]$ be the collection of all restriction of polynomials on $[a, b]$. It forms a subspace of $C[a, b]$. Taking the sequence $h_n(x)$ given by

$$h_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!},$$

$\{h_n\}$ is a Cauchy sequence in $P[a, b]$ which converges to e^x. As e^x is not a polynomial, $P[a, b]$ is not a complete subset of $C[a, b]$.

Proposition 2.15. *Let (X, d) be a metric space.*

(a) Every closed set in X is complete provided X is complete.

(b) Every complete set in X is closed.

(c) Every compact set in X is complete.

Proof. (a) Let (X, d) be complete and E a closed subset of X. Every Cauchy sequence $\{x_n\}$ in E is also a Cauchy sequence in X. By the completeness of X, there is some x in X to which $\{x_n\}$ converges. However, as E is closed, x also belongs to E. So every Cauchy sequence in E has a limit in E.

(b) Let $E \subset X$ be complete and $\{x_n\}$ a sequence converging to some x in X. Since every convergent sequence is a Cauchy sequence, $\{x_n\}$ must converge to some z in E. By the uniqueness of limit, we must have $x = z \in C$, so C is closed.

(c) Let $\{x_n\}$ be a Cauchy sequence in the compact set K. By compactness, there is a subsequence $\{x_{n+j}\}$ converging to some x in X. As every compact set is also closed, x belongs to K. For $\varepsilon > 0$, there exists some n_0 such that $|x_n - x_m| < \varepsilon/2$, for all $n, m \geq n_0$ and $|x_{n_j} - x| \leq \varepsilon/2$, for $n_j \geq n_0$, it follows that

$$|x_n - x| \leq |x_n - x_{n_j}| + |x_{n_j} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for all $n \geq n_0$, that is, $\{x_n\}$ converges to x in K. $\square$

To obtain a typical non-complete set, we consider the interval $[0, 1]$ in $\mathbb{R}$ which is complete and, in fact, compact. Take away one point z from it to form $E = [a, b] \setminus \{z\}$. E is not complete, since every sequence in E converging to z is a Cauchy sequence which does not converge in E. In general, you may think of sets with "holes" being non-complete ones. Now, given a non-complete metric space, can we make it into a complete metric space by filling out all the holes? The answer turns out to affirmative. We can always enlarge a non-complete metric space into a complete one by putting in some ideal points. The process of achieving this goal was long invented by Cantor (1845–1918) in his construction of the real numbers from rational numbers. We start with some formalities.

A metric space (X, d) is called **embedded** in (Y, ρ) if there is a mapping $\Phi : X \to Y$ such that $d(x, y) = \rho(\Phi(x), \Phi(y))$. The mapping Φ is sometimes called a **metric preserving** map. Note that it must be 1-1 and continuous. We call the metric space (Y, ρ) a **completion** of (X, d) if (X, d) is embedded in (Y, ρ) and $\overline{\Phi(X)} = Y$. The latter condition is a minimality condition; (X, d) is enlarged merely to accommodate those ideal points to make the space complete.

Theorem 2.16. *Every metric space has a completion.*

Before the proof we briefly describe the idea. When (X, d) is not complete, we need to invent ideal points and add them to X to make it complete. The idea goes back to Cantor's construction of the real numbers from rational numbers. Suppose now we have only rational numbers and we want to add irrationals. First we identify $\mathbb{Q}$ with a proper subset in a larger set as follows. Let $\mathcal{C}$ be the collection of all Cauchy sequences of rational numbers. Every point in $\mathcal{C}$ is of the form $(x_1, x_2, \cdots)$ where $\{x_n\}, x_n \in \mathbb{Q}$, forms a Cauchy sequence. A rational number x is identified with the constant sequence $(x, x, x, \dots)$ or any Cauchy sequence which converges to x. For instance, 1 is identified with $(1, 1, 1, \dots)$, $(0.9, 0.99, 0.999, \dots)$ or $(1.01, 1.001, 1.0001, \dots)$. Clearly, there are Cauchy sequences which cannot be identified with rational numbers. For instance, there is no rational number corresponding to $(3, 3.1, 3.14, 3.141, 3.1415, \dots)$, as we know, its correspondent should be the irrational number π. Similar situation holds for the sequence $(1, 1.4, 1.41, 1.414, \cdots)$ which should correspond to $\sqrt{2}$. Since the correspondence is not injective, we make it into one by introducing an equivalence relation on $\mathcal{C}$ Indeed, $\{x_n\}$ and $\{y_n\}$ are said to be equivalent if $|x_n - y_n| \to 0$ as $n \to \infty$. The equivalence relation $\sim$ forms the quotient $\mathcal{C}/\sim$ which is denoted by $\widetilde{\mathcal{C}}$. Then $x \mapsto \widetilde{x}$ sends $\mathbb{Q}$ injectively into $\widetilde{\mathcal{C}}$. It can be shown that $\widetilde{\mathcal{C}}$ carries the structure of the real numbers. In particular, those points not in the image of $\mathbb{Q}$ are exactly all irrational numbers. Now, for a metric space the situation is similar. We let $\widetilde{\mathcal{C}}$ be the quotient space of all Cauchy sequence in X under the relation $\{x_n\} \sim \{y_n\}$ if and only if $d(x_n, y_n) \to 0$. Define $\widetilde{d}(\widetilde{x}, \widetilde{y}) = \lim_{n\to\infty} d(x_n, y_n)$, for $x \in \widetilde{x}$, $y \in \widetilde{y}$. We have the embedding $(X, d) \to (\widetilde{X}, \widetilde{d})$, and we can further show that it is a completion of (X, d).

The following proof is for optional reading. In the exercise a simpler but less instructive proof of this theorem can be found.

Proof of Theorem 2.16. Let $\mathcal{C}$ be the collection of all Cauchy sequences in (M, d). We introduce a relation $\sim$ on $\mathcal{C}$ by $x \sim y$ if and only if $d(x_n, y_n) \to 0$ as $n \to \infty$. It is routine to verify that $\sim$ is an equivalence relation on $\mathcal{C}$. Let $\widetilde{X} = \mathcal{C}/\sim$ and define a map: $\widetilde{X} \times \widetilde{X} \mapsto [0, \infty)$ by

$$\widetilde{d}(\widetilde{x}, \widetilde{y}) = \lim_{n\to\infty} d(x_n, y_n)$$

where $x = (x_1, x_2, x_3, \cdots)$ and $y = (y_1, y_2, y_3, \cdots)$ are respective representatives of $\widetilde{x}$ and $\widetilde{y}$. We note that the limit in the definition always exists: For

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

and, after switching m and n,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_m, y_n).$$

As x and y are Cauchy sequences, $d(x_n, x_m)$ and $d(y_m, y_n) \to 0$ as $n, m \to \infty$, and so $\{d(x_n, y_n)\}$ is a Cauchy sequence of real numbers.

Step 1. (well-definedness of $\widetilde{d}$) To show that $\widetilde{d}(\widetilde{x}, \widetilde{y})$ is independent of their representatives, let $x \sim x'$ and $y \sim y'$. We have

$$d(x_n, y_n) \leq d(x_n, x_n') + d(x_n', y_n') + d(y_n', y_n).$$

After switching x and x', and y and y',

$$|d(x_n, y_n) - d(x_n', y_n')| \leq d(x_n, x_n') + d(y_n, y_n').$$

As $x \sim x'$ and $y \sim y'$, the right hand side of this inequality tends to 0 as $n \to \infty$. Hence $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n', y_n')$.

Step 2. ($\widetilde{d}$ is a metric). Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ represent $\widetilde{x}, \widetilde{y}$ and $\widetilde{z}$ respectively. We have

$$\begin{aligned}
\widetilde{d}(\widetilde{x}, \widetilde{z}) &= \lim_{n\to\infty} \big(d(x_n, z_n) \\
&\leq \lim_{n\to\infty} \big(d(x_n, y_n) + d(y_n, z_n)\big) \\
&= \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) \\
&= \widetilde{d}(\widetilde{x}, \widetilde{y}) + \widetilde{d}(\widetilde{y}, \widetilde{z})
\end{aligned}$$

Step 3. We claim that there is a metric preserving map $\Phi : X \mapsto \widetilde{X}$ satisfying $\overline{\Phi(X)} = \widetilde{X}$.

Given any x in X, the "constant sequence" $(x, x, x, \cdots)$ is clearly a Cauchy sequence. Let $\widetilde{x}$ be its equivalence class in $\mathcal{C}$. Then $\Phi x = \widetilde{x}$ defines a map from X to $\widetilde{X}$. Clearly

$$\widetilde{d}(\Phi(x), \Phi(y)) = \lim_{n\to\infty} d(x_n, y_n) = d(x, y)$$

since $x_n = x$ and $y_n = y$ for all n, so Φ is metric preserving and it is injective in particular.

To show that the closure of $\Phi(X)$ is $\widetilde{X}$, we observe that any $\widetilde{x}$ in $\widetilde{X}$ is represented by a Cauchy sequence $x = (x_1, x_2, x_3, \cdots)$. Consider the constant sequence $x^n = (x_n, x_n, x_n, \cdots)$ in $\Phi(X)$. We have

$$\widetilde{d}(\widetilde{x}, \widetilde{x_n}) = \lim_{m\to\infty} d(x_m, x_n).$$

Given $\varepsilon > 0$, there exists an n_0 such that $d(x_m, x_n) < \varepsilon/2$ for all $m, n \geq n_0$. Hence $\widetilde{d}(\widetilde{x}, \widetilde{x_n}) = \lim_{m\to\infty} d(x_m, x_n) < \varepsilon$ for $n \geq n_0$. That is $\widetilde{x^n} \to \widetilde{x}$ as $n \to \infty$, so the closure of $\Phi(M)$ is precisely M.

Step 4. We claim that $(\widetilde{X}, \widetilde{d})$ is a complete metric space. Let $\{\widetilde{x^n}\}$ be a Cauchy sequence in $\widetilde{X}$. As $\overline{\Phi(X)}$ is equal to $\widetilde{M}$, for each n we can find a $\widetilde{y}$ in $\Phi(X)$ such that

$$\widetilde{d}(\widetilde{x^n}, \widetilde{y^n}) < \frac{1}{n} .$$

So $\{\widetilde{y^n}\}$ is also a Cauchy sequence in $\widetilde{d}$. Let y_n be the point in X so that $y^n = (y_n, y_n, y_n, \cdots)$ represents $\widetilde{y^n}$. Since Φ is metric preserving, and $\{\widetilde{y^n}\}$ is a Cauchy sequence in $\widetilde{d}$, $\{y_n\}$ is a Cauchy sequence in X. Let $(y_1, y_2, y_3, \cdots) \in \widetilde{y}$ in $\widetilde{X}$. We claim that $\widetilde{y} = \lim_{n\to\infty} \widetilde{x^n}$ in $\widetilde{X}$. For, we have

$$\begin{aligned}
\widetilde{d}(\widetilde{x^n}, \widetilde{y}) &\leq \widetilde{d}(\widetilde{x^n}, \widetilde{y^n}) + \widetilde{d}(\widetilde{y^n}, \widetilde{y}) \\
&\leq \frac{1}{n} + \lim_{m\to\infty} d(y_n, y_m) \to 0
\end{aligned}$$

as $n \to \infty$. We have shown that $\widetilde{d}$ is a complete metric on $\widetilde{X}$. $\square$

Completion of a metric space is unique once we have clarified the meaning of uniqueness. Indeed, call two metric spaces (X, d) and (X', d') **isometric** if there exists a bijective embedding from (X, d) onto (X', d'). Since a metric preserving map is always one-to-one, the inverse of of this mapping exists and is a metric preserving mapping from (X', d') to (X, d). So two spaces are isometric provided there is a metric preserving map from one onto the other. Two metric spaces will be regarded as the same if they are isometric, since then they cannot be distinguish after identifying a point in X with its image in X' under the metric preserving mapping. With this understanding, the completion of a metric space is unique in the following sense: If (Y, ρ) and (Y', ρ') are two completions of (X, d), then (Y, ρ) and (Y', ρ') are isometric. We will not go into the proof of this fact, but instead leave it to the interested reader. In any case, now it makes sense to use "the completion" of X to replace "a completion" of X.

2.5 The Contraction Mapping Principle

Solving an equation $f(x) = 0$, where f is a function from $\mathbb{R}^n$ to itself frequently comes up in application. This problem can be turned into a problem for fixed points. Literally, a fixed point of a mapping is a point which becomes unchanged under this mapping. By introducing the function $g(x) = f(x) + x$, solving the equation $f(x) = 0$ is equivalent to finding a fixed point for g. This general observation underlines the importance of finding fixed points. In this section we prove the contraction mapping principle, one of the oldest fixed point theorems and perhaps the most well-known one. As we will see, it has a wide range of applications.

A map $T : (X, d) \to (X, d)$ is called a **contraction** if there is a constant $\gamma \in (0, 1)$ such that $d(Tx, Ty) \le \gamma d(x, y)$, $\forall x, y \in X$. A point x is called a **fixed point** of T if $Tx = x$. Usually we write Tx instead of $T(x)$.

Theorem 2.17 (**Contraction Mapping Principle**). *Every contraction in a complete metric space admit a unique fixed point.*

Proof. Let T be a contraction in the complete metric space (X, d). Pick an arbitrary $x_0 \in X$ and define a sequence $\{x_n\}$ by setting $x_n = Tx_{n-1} = T^n x_0$, $\forall n \ge 1$. We claim that $\{x_n\}$ forms a Cauchy sequence in X. First of all, by iteration we have

$$\begin{aligned} d(T^n x_0, T^{n-1} x_0) &\le \gamma d(T^{n-1} x_0, T^{n-2} x_0) \\ &\cdot \\ &\cdot \\ &\le \gamma^{n-1} d(Tx_0, x_0). \end{aligned} \tag{2.1}$$

Next, for $n \ge N$ where N is to be specified in a moment,

$$\begin{aligned} d(x_n, x_N) &= d(T^n x_0, T^N x_0) \\ &\le \gamma d(T^{n-1} x_0, T^{N-1} x_0) \\ &\le \gamma^N d(T^{n-N} x_0, x_0). \end{aligned}$$

By the triangle inequality and (2.1),

$$\begin{aligned} d(x_n, x_N) &\le \gamma^N \sum_{j=1}^{n-N} d(T^{n-N-j+1} x_0, T^{n-N-j} x_0) \\ &\le \gamma^N \sum_{j=1}^{n-N} \gamma^{n-N-j} d(Tx_0, x_0) \\ &< \frac{d(Tx_0, x_0)}{1-\gamma} \gamma^N. \end{aligned} \tag{2.2}$$

For $\varepsilon > 0$, choose N so large that $d(Tx_0, x_0)\gamma^N/(1-\gamma) < \varepsilon/2$. Then for $n, m \ge N$,

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_N) + d(x_N, x_m) \\ &< \frac{2d(Tx_0, x_0)}{1-\gamma} \gamma^N \\ &< \varepsilon, \end{aligned}$$

thus $\{x_n\}$ forms a Cauchy sequence. As X is complete, $x = \lim_{n\to\infty} x_n$ exists. By the continuity of T, $\lim_{n\to\infty} Tx_n = Tx$. But on the other hand, $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x$. We conclude that $Tx = x$.

Suppose there is another fixed point $y \in X$. From

$$\begin{aligned} d(x,y) &= d(Tx,Ty) \\ &\leq \gamma d(x,y), \end{aligned}$$

and $\gamma \in (0,1)$, we conclude that $d(x,y) = 0$, i.e., $x = y$. □

Incidentally, we point out that this proof is a constructive one. It tells you how to find the fixed point starting from an arbitrary point. In fact, letting $n \to \infty$ in (2.2) and then replacing N by n, we obtain an error estimate between the fixed point and the approximating sequence $\{x_n\}$:

$$d(x, x_n) \leq \frac{d(Tx_0, x_0)}{1-\gamma}\gamma^n, \quad n \geq 1.$$

Example 2.18. Let us take X to be $\mathbb{R}$. Then T is nothing but a real-valued function on $\mathbb{R}$. Denoting the identity map $x \mapsto x$ by I. A point on the graph of T is given by (x, Tx) and a point on the graph of I is (x,x). So every intersection point of both graphs $(x,Tx) = (x,x)$ is a fixed point of T. From this point of view we can see functions may or may not have fixed points. For instance, the function $Tx = x + e^x$ does not have any fixed point. By drawing graphs one is convinced that there are functions with graphs lying below the diagonal line and yet whose slope is always less than one but tends to 1 at infinity (see exercise for a concrete one). It shows the necessity of $\gamma \in (0,1)$. On the other hand, functions like $Sx = x(x-1)(x+2)$ whose graph intersects the diagonal line three times, so it has three fixed points. The insight of Banach's fixed point theorem is to single out a class of functions which admits one and only one fixed point. The contractive condition can be expressed as

$$\left|\frac{Tx - Ty}{x - y}\right| < \gamma, \quad \forall x, y.$$

It means that the slope of T is always bounded by $\gamma \in (0,1)$. Let (x,Tx) be a point of the graph of T and consider the cone emitting from this point bounded by two lines of slopes $\pm\gamma$. When T is a contraction, it is clear that its graph lies within this cone. A moment's reflection tells us that it must hit the diagonal line exactly once.

Example 2.19. Let $f : [0,1] \to [0,1]$ be a continuously differentiable function satisfying $|f'(x)| < 1$ on $[0,1]$. We claim that f admits a fixed point. For, by the mean value theorem, for $x, y \in [0,1]$ there exists some $z \in (0,1)$ such that $f(y) - f(x) = f'(z)(y-x)$. Therefore,

$$\begin{aligned} |f(y) - f(x)| &= |f'(z)||y - x| \\ &\leq \gamma|y - x|, \end{aligned}$$

where $\gamma = \sup_{t\in[0,1]} |f'(t)| < 1$ (Why?). We see that f is a contraction. By the contraction mapping principle, it has a fixed point. In fact, by using the mean-value theorem one can show that *every continuous function* from $[0,1]$ to itself admits at least one fixed point. This is a general fact. According to Brouwer's fixed point theorem, every continuous maps from a compact convex set in $\mathbb{R}^n$ to itself admits one fixed point. This theorem surely includes the present case. However, when the set has "non-trivial topology", fixed points may not exist. For instance, take X to be $A = \{(x,y) : \ 1 \le x^2+y^2 \le 4\}$ and T to be a rotation. It is clear that T has no fixed point in A. This is due to the topology of A, namely, it has a hole.

2.6 Picard-Lindelöf Theorem for Differential Equations

In this section we discuss the fundamental existence and uniqueness theorem for differential equations. I assume that you learned the skills of solving ordinary differential equations in MATH3270 so we will focus on the theoretical aspects.

Most differential equations cannot be solved explicitly, in other words, they cannot be expressed as the composition of elementary functions. Nevertheless, there are two exceptional classes which come up very often. Let us review them before going into the theory. The first one is linear equation.

$$\frac{dy}{dx} = a(x)y + b(x),$$

where a and b are continuous functions defined on some interval I. The general solution of this linear equation is given by the formula

$$y(x) = e^{A(x)}\left(y_0 + \int_{x_0}^x e^{-A(t)}b(t)dt\right),$$

where $x_0 \in I, y_0 \in \mathbb{R}$, are fixed and $A(x) = \int_{x_0}^x a(t)dt$. The second class is the so-called separable equation

$$\frac{dy}{dx} = \frac{f(x)}{g(y)},$$

where f and $g \neq 0$ are continuous functions on intervals I and J respectively. Then the solution can be obtained by an integration

$$\int_{y_0}^y g(t)dt = \int_{x_0}^x f(s)ds, \qquad x_0 \in I, \ y_0 \in J.$$

The resulting relation, written as $G(y) = F(x)$, can be converted into $y = G^{-1}F(x)$, a solution to the equation as immediately verified by the chain rule. These two classes of equations are sufficient for our purpose. More interesting explicitly solvable equations can be found in texts on ODE's.

Numerous problems in natural sciences and engineering led to the initial value problem of differential equations. Let f be a function defined in the rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ for $(x_0, y_0) \in \mathbb{R}^2$ and $a, b > 0$. We consider the initial value problem or Cauchy problem

$$\begin{cases} \dfrac{dy}{dx} = f(x, y), \\ y(x_0) = y_0. \end{cases} \tag{2.3}$$

To solve the Cauchy problem it means to find a function $y(x)$ defined in a perhaps smaller rectangle, that is, $y : [x_0 - a', x_0 + a'] \to [y_0 - b, y_0 + b]$, which is differentiable and satisfies $y(x_0) = y_0$ and $y'(x) = f(x, y(x))$, $\forall x \in [x_0 - a', x_0 + a']$, for some $0 < a' \leq a$. In general, no matter how nice f is, we do not expect there is always a solution on the entire $[x_0 - a, x_0 + a]$. Let us look at the following example.

Example 2.20. Consider the Cauchy problem

$$\begin{cases} \dfrac{dy}{dx} = 1 + y^2, \\ y(0) = 0. \end{cases}$$

The function $f(x, y) = 1 + y^2$ is smooth on $[-a, a] \times [-b, b]$ for every $a, b > 0$. However, a solution, as one can verify immediately, is given by $y(x) = \tan x$ which is only defined on $(-\pi/2, \pi/2)$. It shows that even when f is very nice, a' could be strictly less than a.

The Picard-Lindelöf theorem, sometimes referred to as the fundamental theorem of existence and uniqueness of differential equations, gives a clean condition on f ensuring the unique solvability of the Cauchy problem (2.3). This condition imposes a further regularity condition on f reminding what we did in the convergence of Fourier series. Specifically, a function f defined in R satisfies the **Lipschitz condition** if there exists $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L\,|y_1 - y_2|\,, \quad \forall (x, y_i) \in R \equiv \in [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b],\ i = 1, 2.$$

Note that in particular means for each fixed x, f is Lipschitz continuous in y. The least constant L^* satisfies this relation, given by

$$L^* = \inf\{L : \quad \text{The relation above holds for } L\},$$

is called the **Lipschitz constant** of f. Some authors call any L satisfies the inequality above a Lipschitz constant. Clearly, we still have

$$|f(x, y_1) - f(x, y_2)| \leq L^*\,|y_1 - y_2|\,, \quad \forall (x, y_i) \in R,\ i = 1, 2.$$

Not all continuous functions satisfy the Lipschitz condition. An example is given by the function $f(x, y) = xy^{1/2}$ is continuous. I let you verify that it does not satisfy the Lipschitz condition on any rectangle containing the origin.

A C^1-function $f(x, y)$ in a rectangle automatically satisfies the Lipschitz condition. For, by the mean-value theorem, for some z lying on the segment between y_1 and y_2,

$$f(x, y_2) - f(x, y_1) = \frac{\partial f}{\partial y}(x, z)(y_2 - y_1).$$

Letting

$$L = \max\left\{ \left|\frac{\partial f}{\partial y}(x, y)\right| : \ (x, y) \in R \right\},$$

(L is finite because $\partial f/\partial y$ is continuous on R and hence bounded), we have

$$|f(x, y_2) - f(x, y_1)| \leq L|y_2 - y_1|, \quad \forall (x, y_i) \in R, \ i = 1, 2.$$

In practise, many f one encounters are C^1 in their domains of definition.

Theorem 2.18 (**Picard-Lindelöf Theorem**). *Consider (2.3) where $f \in C(R)$ satisfies the Lipschitz condition on $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$. There exist $a' \in (0, a)$ and $y \in C^1[x_0 - a', x_0 + a']$, $y_0 - b \leq y(x) \leq y_0 + b$ for all $x \in [x_0 - a', x_0 + a']$, solving (2.3).*

From the proof one will see that a' can be taken to be any number satisfying

$$0 < a' < \min\{a, \frac{b}{M}, \frac{1}{L^*}\},$$

where $M = \sup\{|f(x, y)| : \ (x, y) \in R\}$.

We first convert (2.3) into a single integral equation.

Proposition 2.19. *Setting as in Theorem 2.17, every solution y of (2.3) from $[x_0 - a', x_0 + a']$ to $[y_0 - b, y_0 + b]$ satisfies the equation*

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\, dx. \tag{2.4}$$

Proof. When y satisfies $y'(x) = f(x, y(x))$ and $y(x_0) = y_0$, (2.4) is a direct consequence of the fundamental theorem of calculus (first form). Conversely, when $y(x)$ is continuous on $[x_0 - a', x_0 + a']$, $f(x, y(x))$ is also continuous on the same interval. By the fundamental theorem of calculus (second form), the left hand side of (2.4) is continuously differentiable on $[x_0 - a', x_0 + a']$ and solves (2.3). □

Note that in this proposition we do not need the Lipschitz condition; only the continuity of f is needed.

Proof of Theorem 2.17. Instead of solving (2.3) directly, we look for a solution of (2.4). We will work on the metric space $X = \{f \in C[x_0 - a', x_0 + a'] : \ f(x) \in [y_0 - b, y_0 + b], f(x_0) = y_0\}$ with the uniform metric. It is easily verified that it is a complete metric space. The number a' will be specified below.

We are going to define a contraction on X. Indeed, for $y \in X$, define T by

$$(Ty)(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\, dx.$$

First of all, for every $y \in X$, it is clear that $f(x, y(x))$ is well-defined and $Ty \in C[x_0 - a', x_0 + a']$. To show that it is in X, we need to verify $y_0 - b \le (Ty)(x) \le y_0 + b$ for all $x \in [x_0 - a', x_0 + a']$. We claim this holds if we choose a' satisfying $a' \le b/M$, $M = \sup\{|f(x, y)| : (x, y) \in R\}$. For,

$$\begin{aligned} |(Ty)(x) - y_0| &= \left| \int_{x_0}^{x} f(x, y(x))\, dx \right| \\ &\le M\, |x - x_0| \\ &\le Ma' \\ &\le b. \end{aligned}$$

Next, we claim T is a contraction on X when a' is further restricted to $a' \le 1/(2L^*)$, where L^* is the Lipschitz constant for f. For,

$$\begin{aligned} |(Ty_2 - Ty_1)(x)| &= \left| \int_{x_0}^{x} f(x, y_2(x)) - f(x, y_1(x))\, dx \right| \\ &\le \int_{x_0}^{x} \big| f(x, y_2(x)) - f(x, y_1(x)) \big|\, dx \\ &\le L^* \int_{x_0}^{x} |y_2(x) - y_1(x)|\, dx \\ &\le L^* \sup_{x \in I} |y_2(x) - y_1(x)|\, |x - x_0| \\ &\le L^* a' \sup_{x \in I} |y_2(x) - y_1(x)| \\ &\le \frac{1}{2} \sup_{x \in I} |y_2(x) - y_1(x)|, \end{aligned}$$

where $I = [x_0 - a', x_0 + a']$. It follows that

$$d_\infty(Ty_2, Ty_1) \le \frac{1}{2} d_\infty(y_2, y_1)$$

where d_∞ is the uniform metric $d_\infty(f, g) \equiv \|f - g\|_\infty$ for $f, g \in C[x_0 - a', x_0 + a']$. Now we can apply the contraction mapping principle to conclude that $Ty = y$ for some y, and y solves (2.3). We have shown that (2.3) admits a solution in $[x_0 - a', x_0 + a']$ where a' can be chosen to be $\min\{b/M, 1/(2L^*)\}$. Apparently the same conclusion holds when 2 is replaced by any number greater than 1 in this expression. □

We formulate a uniqueness and extension result in the following proposition.

Proposition 2.20. *Let y_1 and y_2 be solutions to (2.3) on closed and bounded intervals I_1 and I_2 containing x_0 in its interior respectively. Under the Lipschitz condition on f, y_1 and y_2 coincide on $I_1 \cap I_2$. Therefore, the function y which is equal to y_1 on I_1 and y_2 on I_2 is a solution to (2.3) on $I_1 \cup I_2$.*

Proof. Let $J = I_1 \cap I_2 \equiv [\alpha, \beta]$ and set $z = \sup\{x : \ y_1 \equiv y_2 \text{ on } [x_0, x]\}$. We claim that $z = \beta$. For, if $z < \beta$, by continuity $y_1(z) = y_2(z)$. For $x \in [\alpha, \beta]$, we have

$$\begin{aligned} |y_1(x) - y_2(x)| &= \left| \int_z^x f(t, y_1(t)) - f(t, y_2(t))\, dt \right| \\ &\leq L^* \int_z^x |y_1(t) - y_2(t)|\ dt. \end{aligned}$$

Let $x \in J = [z-1/(2L^*), z+1/(2L^*)] \cap [\alpha, \beta]$ and $|y_1(x_1) - y_2(x_1)| = \max_{x \in J} |y_1(x) - y_2(x)|$. Then

$$\begin{aligned} \max_J |y_1(x) - y_2(x)| &= |y_1(x_1) - y_2(x_1)| \\ &\leq L^* \max_J |y_1(x) - y_2(x)|\, |x_1 - z| \\ &\leq \frac{1}{2} \max_J |y_1(x) - y_2(x)|\,, \end{aligned}$$

which forces $y_1 \equiv y_2$ on $[z - 1/(2L^*), z + 1/(2L^*)] \cap [\alpha, \beta]$. It means that y_1 and y_2 coincide on $[x_0, \min\{z + 1/2L^*, \beta\}]$, contradicting the definition of z. Hence y_1 and y_2 must coincide on $[x_0, \beta]$. A similar argument shows that they coincide on $[\alpha, x_0]$. $\square$

A consequence of Theorem 2.17 and Proposition 2.19 is the existence of a maximal solution.

Theorem 2.21. *Consider (2.3) where f is a continuous function on the open set G such that it satisfies the Lipschitz condition on every compact subset of G. Then there exists a solution y^* to (2.3) defined on some (α, β) satisfying*

(a) Whenever y is a solution of (2.3) on some interval I, $I \subset (\alpha, \beta)$ and $y = y^*$ on I .

(b) If β is finite, the solution escapes from every compact subset of G eventually. Similar results holds at α.

"The solution escapes from every compact subset of G eventually" means, for each compact $K \subset G$, there exists a small $\delta > 0$ such that $(x, y^*(x)) \in G \setminus K$ for $x \in [\beta - \delta, \beta)$. When G is $\mathbb{R}^2$, it means that y^* either tends to ∞ or $-\infty$ as x approaches β when β is finite. When β is infinite, the solution could tend to positive or negative infinity or oscillate up and down infinitely as x goes to ∞. In contrast, when β is finite, the solution could either goes to ∞ or $-\infty$ approaching β. The case of oscillation is excluded.

The Lipschitz condition on f sounds a little complicated. We could replace it by a simpler condition. Indeed, the subsets

$$K_n = \{x \in G : \ \mathrm{dist}(x, \partial G) \geq 1/n\} \cap \overline{B_n(0)}, \ n \geq 1,$$

are compact and $G = \bigcup_{n=1}^{\infty} K_n$. Clearly every compact subset is contained in some K_n for sufficiently large n. With this understanding, the Lipschitz condition on f may be recast as, there exist $L_n, n \geq 1$, such that

$$\big|f(x, y_2) - f(x, y_1)\big| \leq L_n |y_1 - y_1|, \quad \forall y_1, y_2 \in K_n.$$

In view of this theorem, it is legal to call this *maximal solution the* solution of (2.3) and the interval (α, β) the maximal interval of existence.

Proof. Let $\mathcal{I}$ be the collection of all closed, bounded intervals I containing x_0 over which a solution of (2.3) exists and let I^* be the union of the intervals in $\mathcal{I}$. Clearly I^* is again an interval, denote its left and right endpoints by α and β respectively. By Proposition 2.20 there is a solution y^* of (2.3) defined on (α, β). When β is finite, let us show that the solution escapes from every compact subset eventually. Let K be a compact subset of G and suppose on the contrary that there exists $\{x_k\} \subset (\alpha, \beta), \ x_k \to \beta$, but $(x_k, y^*(x_k)) \in K$ for all k. By compactness, we may assume $y^*(x_k)$ converges to some z in K (after passing to a subsequence if necessary). Since $\mathrm{dist}((\beta, z), \partial G) > 0$, we can find a rectangle $[\beta - r, \beta + r] \times [z - \rho, z + \rho]$ inside G. Then, as this is a compact subset, f satisfies the Lipschitz condition on this rectangle. By Theorem 2.18, we could use $(x_k, y^*(x_k))$ as the initial data to solve (2.3). The range of this solution would be some interval $[x_k - r', x_k + r']$ where r' is independent of k. Since x_k approaches β, for large k $\beta \in [x_k - r', x_k + r']$. But then by Proposition 2.20, the solution y^* can be extended beyond β, contraction holds. We conclude that the solution must escape from any compact subset eventually. A similar argument applies to the left endpoint α.

□

Example 2.21. Consider

$$f(x, y) = \frac{x}{1 - y}, \qquad (x, y) \in G \equiv (-\infty, \infty) \times (-\infty, 1)$$

and $x_0 = y_0 = 0$ in (2.3). Since $f \in C^1(G)$, the setting of Theorem 2.1 is satisfied. This equation is separable and the solution is readily found to be

$$y(x) = 1 - \sqrt{1 - x^2}.$$

So the maximal interval of existence is given by $(-1, 1)$. As $x \to \pm 1$, $(x, y(x))$ hits the horizontal line $y = 1$ as asserted by Theorem 2.21.

We point out that the existence part of Picard-Lindelöf theorem still holds without the Lipschitz condition. We will prove this in the next chapter. However, the solution may not be unique.

Example 2.22. Consider the Cauchy problem $y' = |y|^\alpha$, $\alpha \in (0,1)$, $y(0) = 0$. The function $f(x) = |x|^\alpha$ is Hölder continuous but not Lipschitz continuous. While $y_1 \equiv 0$ is a solution,

$$y_2 = (1-\alpha)^{\frac{1}{1-\alpha}} |x|^{\frac{1}{1-\alpha}}$$

is also a solution. In fact, there are infinitely many solutions! Can you write them down?

Theorem 2.18, Propositions 2.20 and 2.21 are valid for systems of differential equations. Without making things too clumsy, we put all results in a single theorem. First of all, the Cauchy problem for systems of differential equations is

$$\begin{cases} \dfrac{dy_j}{dx} = f_j(x, y_1, y_2, \cdots, y_N), \\ y_j(x_0) = y_{j0}, \end{cases}$$

where $j = 1, 2, \cdots, N$. By setting $y = (y_1, y_2, \cdots, y_N)$ and $f = (f_1, f_2, \cdots, f_N)$, we can express it as in (2.3) but now both y and f are vectors.

Essentially following the same arguments as the case of a single equation, we have

Theorem 2.22 (**Picard-Lindelöf Theorem for Systems**). *Consider (2.3) where f satisfies a Lipschitz condition on $R = [x_0 - a, x_0 + a] \times J$.*

(a) There exist $a' \in (0, a)$ and a unique C^1-function y on $[x_0 - a', x_0 + a']$ solving (2.3).

(b) Let y_1 and y_1 be two solutions of (2.3) on closed, bounded intervals I_1 and I_2 respectively. The y_1 and y_2 coincide on $I_1 \cap I_2$.

Here J stands for $\prod_{j=1}^n [y_{0j} - b_j, y_{0j} + b_j]$ and the Lipschitz condition on f should be interpreted as

$$d_2(f(x, y_1), f(x, y_2)) \le L d_2(y_1, y_2), \quad \forall x \in [x_0 - a, x_0 + a].$$

There is also a version on systems corresponding to Theorem 2.21. We will omit it.

We remind you that there is a standard way to convert the Cauchy problem for higher order differential equation ($m \ge 2$)

$$\begin{cases} y^{(m)} = f(x, y, y', \cdots, y^{(m-1)}), \\ y(x_0) = y_0,\ y'(x_0) = y_1, \cdots, y^{(m-1)}(x) = y_{m-1}, \end{cases}$$

into a system of first order differential equations. As a result, we also have a corresponding Picard-Lindelöf theorem for higher order differential equations as well as the existence of a maximal solution. I will let you formulate these results.

Comments on Chapter 2. A topology on a set X is a collection of sets τ consisting the empty and X itself which is closed under arbitrary union and finite intersection. Each set in τ is called an open set. The pair (X, τ) is called a topological space. From Proposition 2.2 we see that the collection of all open sets in a metric space (X, d) forms a topology on X. This is the topological space induced by the metric. Metric spaces constitute a large class of topological spaces, but not every topological space comes from a metric. However, from the discussions in Section 2 we know that continuity can be defined solely in terms of open sets. It follows that continuity can be defined for topological spaces, and this is crucial for many further developments. In the past, metric spaces were covered in Introductory Topology. Feeling that the notion of a metric space should be learned by every math major, we move it here. This restructure of curriculum also leaves room for more algebraic topology in Introductory Topology.

Wiki gives a nice summary of metric spaces under "metric space".

Many theorems in finite dimensional space are extended to infinite dimensional normed spaces when the underlying closed, bounded set is replaced by a compact set. Thus it is extremely important to study compact sets in a metric space. We will study compact sets in $C[a, b]$ in Chapter 3. A theorem of Arzela-Ascoli provides a complete characterization of compact sets in this space.

There are two popular constructions of the real number system, Dedekind cuts and Cantor's Cauchy sequences. Although the number system is fundamental in mathematics, we did not pay much attention to its rigorous construction. It is too dry and lengthy to be included in MATH2050 . Indeed, there are two sophisticate steps in the construction of real numbers from nothing, namely, the construction of the natural numbers by Peano's axioms and the construction of real numbers from rational numbers. Other steps are much easier. Cantor's construction of the irrationals from the rationals is very much like the proof of Theorem 2.15. You may google under the key words "Peano's axioms, Cantor's construction of the real numbers, Dedekind cuts" for more.

The contraction mapping principle, or Banach fixed point theorem, was found by the Polish mathematician S Banach (1892-1945) in his 1922 doctoral thesis. He is the founder of functional analysis and operator theory. According to P Lax, "During the Second World War, Banach was one of a group of people whose bodies were used by the Nazi occupiers of Poland to breed lice, in an attempt to extract an anti-typhoid serum. He died shortly after the conclusion of the war." The interested reader should look up his biography at Wiki.

An equally famous fixed point theorem is Brouwer's fixed point theorem. It states that every continuous map from a closed ball in $\mathbb{R}^n$ to itself admits at least one fixed point. Here it is not the map but the geometry, or more precisely, the topology of the ball matters. You will learn it in Introductory Topology.

Picard-Lindelöf theorem or the fundamental existence and uniqueness theorem of differential equations was mentioned in Ordinary Differential Equations and now its proof is discussed in details. Of course, the contributors also include Cauchy and Lipschitz. Further results without the Lipschitz condition can be found in Chapter 3. A classic text on ordinary differential equations is "Theory of Ordinary Differential Equations" by Coddington and Levinson. V.I. Arnold's "Ordinary Differential Equations" is also a popular text.

Although metric space is a common topic, I found it difficult to fix upon a single reference book. Rudin's Principles covers some metric spaces, but his attention is mainly on the Euclidean space. Moreover, for a devoted student, this book should have been studied in a previous summer. Finally, I decide to list Dieudonne's old book "Foundation of Modern Analysis" as the only reference. This is the book from which I learned the subject, but it seems a bit out-dated and not easy to follow. Another good reference which is more comprehensible but contains less content is G.F. Simmons "Introduction to Topology and Modern Analysis". The chapters on metric and topological spaces are highly readable.